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The deflection of a simply supported plate induced by piezoelectric actuators

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Abstract

The coupling effects between the mechanical and electric properties of piezoelectric materials have drawn significant attention for their potential applications as sensors and actuators. In this investigation, two piezoelectric actuators are symmetrically embedded in a simply supported plate. Electric voltages with the same amplitude and opposite sign are applied to the two symmetric piezoelectric actuators, resulting in the bending effect on the plate. The bending moment is derived by using the theories of elasticity and piezoelectricity. The analytical solution of the flexural displacement of a simply supported plate subjected to the bending moment is solved by using the plate theory. The analytical solution is compared with the finite element solution to show the validation of present approach. The effects of the size and location of the piezoelectric actuators on the response of a plate are presented through a parametric study.

Keywords: Piezoelectric actuator; Bending moment; Plate theory; Flexural displacement

1. Introduction

Piezoelectric materials with the advantages of quick response, low power consumption and high linearity have drawn much attention in the past decade. Piezoelectric devices are of great interest in structural engineering with applications to shape control, vibration suppression and noise reduction [1]. Smart structures integrated with sensors and actuators have the capability to respond to a changing environment and control the structural movement. Piezoceramics are the most common material used in smart structures and can be surface bonded to existing structures to form an online monitoring system, or embedded in composite structures without significantly changing the structural stiffness system. Bailey and Hubbard [2] developed first adaptive structure using polyvinylidene fluoride (PVDF) film as actuators to control the structural vibration of a cantilever beam. Crawley and de Luis [3] studied a beam with surface bonded and embedded piezoelectric actuators to investigate the load transfer between the actuator and host beam. Huang and Sun [4] studied the load transfer and wave propagation of an anisotropic elastic medium induced by the surface bonded piezoelectric actuator. Dimitriadis et al. [5] used two dimensional patches of piezoelectric material bonded to the surface of a simply supported plate as vibration actuators to excite the selected modes. Shape control is one of the major applications for piezoelectric materials. Koconis et al. [6] controlled the shape of composite plates and shells with piezoelectric actuators. Luo and Tong [7] developed a finite element model to simulate twisting and bending shape control using the orthotropic piezoelectric actuators. Lin and Nien [8] used piezoelectric actu-

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ators to control the deflection and shape of the composite laminates.

Present work investigated the load transfer between the embedded piezoelectric actuators and host structure. The proposed method is an extension of the one dimensional beam embedded with PZT derived by Crawley and de Luis [3] and two dimensional plate surface bonded with PZT derived by Dimitriadis et al. [5]. The model consists of two piezoelectric actuators symmetrically embedded in a simply supported plate subjected to electrical voltage. An analytical expression of the bending moment induced by the piezoelectric actuators was derived by incorporating the theory of elasticity and piezoelectric effect. The bending moment was then applied to the simply supported plate. A closed form solution to the deflection of the plate was obtained by using the plate theory. The analytical solution was validated with the finite element results. The effects of the size and location of the piezoelectric actuators on the response of a plate are presented through a parametric study. The objective of this investigation is to develop an analytical expression of the response of a thin plate excited by the embedded piezoelectric actuators. The feasibility of controlling the deflected shape of the plate is illustrated by placing the actuators at various locations.

2. Load transfer

Consider two piezoelectric actuators symmetrically embedded in a homogeneous plate. The polarized direction is along the z-axis. For an unconstrained thin piezoelectric actuator, equal strains in both x and y directions will be induced when activated by a voltage along the poling direction. The magnitude of the strain can be expressed in terms of the piezoelectric constant d_{31}, applied voltage V and actuator thickness t_{pe}, as follows:

$$(\varepsilon_x)_{pe} = (\varepsilon_y)_{pe} = \varepsilon_{pe} = \frac{d_{31}}{t_{pe}} V \tag{1}$$

where pe and p represent the quantities associated with piezoelectric actuator and host plate, respectively, throughout this paper.

The two actuators are activated by applying a voltage of equal magnitude and opposite sign to the opposing actuators. The opposite directions of the surface tractions at the interfaces between the actuator and plate cause the uniform bending moments along the actuator boundaries. The symmetry of the actuators with respect to the midplane (z = 0) results in no net extension or contraction in the midplane of the plate. In the following derivation, the piezoelectric actuators are assumed to be perfectly bonded to the host plate and in the state of plane stress. This implies the strain continuity across the interfaces.

The forces at the interfaces between the plate and piezoelectric actuators are shown in Fig. 1. $(F_x)_t$ and $(F_x)_b$ represent the forces in the x-direction per unit width (along the y-direction) where the subscripts t and b denote the top and bottom interfaces between the embedded actuator and plate, respectively. Similarly, $(F_y)_t$ and $(F_y)_b$ represent the forces in the y-direction per unit width (along the x-direction). The stresses in the piezoelectric actuator induced by these interfacial forces can be expressed as [3]

$$(\sigma_x)_{pe} = \frac{6\Delta F_x}{t_{pe}^2}(z - z_m) + \frac{\bar{F}_x}{t_{pe}};$$
$$(\sigma_y)_{pe} = \frac{6\Delta F_y}{t_{pe}^2}(z - z_m) + \frac{\bar{F}_y}{t_{pe}} \tag{2}$$

$$\Delta F_x = (F_x)_b - (F_x)_t \;;\; \bar{F}_x = (F_x)_b + (F_x)_t;$$
$$\Delta F_y = (F_y)_b - (F_y)_t \;;\; \bar{F}_y = (F_y)_b + (F_y)_t \tag{3}$$

where z_m is the distance between the two midplanes of the plate and piezoelectric actuator.

Since the two actuators are symmetrically embedded in the plate, the interfacial forces in the top and bottom actuators acting in opposite directions form a couple system. The couple moment per unit width applying to the plate as shown in Fig. 2 can be written as

$$m_x = 2\left[\Delta F_x \frac{t_{pe}}{2} + \bar{F}_x z_m\right];$$
$$m_y = 2\left[\Delta F_y \frac{t_{pe}}{2} + \bar{F}_y z_m\right] \tag{4}$$

The bending stresses in the plate are

$$(\sigma_x)_p = \frac{12(m_x)z}{t_p^3};$$
$$(\sigma_y)_p = \frac{12(m_y)z}{t_p^3} \tag{5}$$

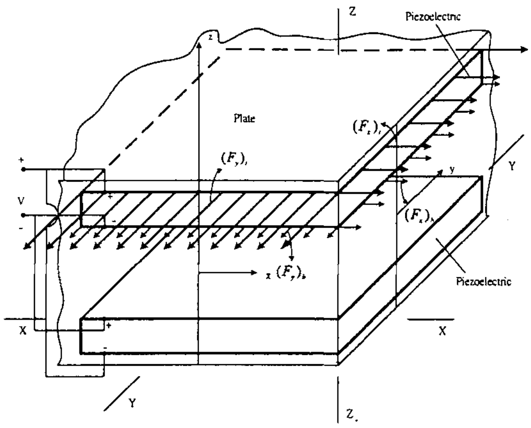

Fig. 1. Forces developed in the interface.

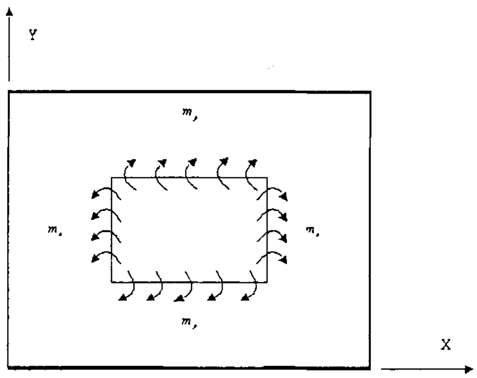

Fig. 2. Bending moment acting on the plate.

Substituting $z = z_m - \frac{t_{pe}}{2}$ and $z = z_m + \frac{t_{pe}}{2}$ into Eq. (2), leads to the stresses in the plate at the bottom and top interfaces as follows.

$$(\sigma_x)_p^b = \frac{12t_{pe}(z_m - t_{pe}/2)\Delta F_x + 24z_m(z_m - t_{pe})\bar{F}_x}{t_p^3} \tag{6a}$$

$$(\sigma_y)_p^b = \frac{12t_{pe}(z_m - t_{pe}/2)\Delta F_y + 24z_m(z_m - t_{pe})\bar{F}_y}{t_p^3} \tag{6b}$$

$$(\sigma_x)_p^t = \frac{12t_{pe}(z_m + t_{pe}/2)\Delta F_x + 24z_m(z_m + t_{pe})\bar{F}_x}{t_p^3} \tag{6c}$$

$$(\sigma_y)_p^t = \frac{12t_{pe}(z_m + t_{pe}/2)\Delta F_y + 24z_m(z_m + t_{pe})\bar{F}_y}{t_p^3} \tag{6d}$$

where the superscripts b and t represent the bottom and top interfaces between the embedded actuator and plate.

Substituting $z = z_m - \frac{t_{pe}}{2}$ and $z = z_m + \frac{t_{pe}}{2}$ into Eq. (5), results to the stresses in the piezoelectric actuator at the bottom and top interfaces

$$(\sigma_x)_{pe}^b = \frac{\bar{F}_x - 3\Delta F_x}{t_{pe}} \ ; \quad (\sigma_y)_{pe}^b = \frac{\bar{F}_y - 3\Delta F_y}{t_{pe}} \ ;$$
$$(\sigma_x)_{pe}^t = \frac{\bar{F}_x + 3\Delta F_x}{t_{pe}} \ ; \quad (\sigma_y)_{pe}^t = \frac{\bar{F}_y + 3\Delta F_y}{t_{pe}} \tag{7}$$

Substituting Eq. (6) into strain-stress relationship, yields to the strains in the plate at the bottom and top interfaces as follows.

$$(\varepsilon_x)_p^b = \frac{12t_{pe}(z_m - t_{pe}/2)(\Delta F_x - \nu_p \Delta F_y) + 24z_m(z_m - t_{pe}/2)(\bar{F}_x - \nu_p \bar{F}_y)}{E_p t_p^3} \tag{8a}$$

$$(\varepsilon_y)_p^b = \frac{12t_{pe}(z_m - t_{pe}/2)(\Delta F_y - \nu_p \Delta F_x) + 24z_m(z_m - t_{pe}/2)(\bar{F}_y - \nu_p \bar{F}_x)}{E_p t_p^3} \tag{8b}$$

$$(\varepsilon_x)_p^t = \frac{12t_{pe}(z_m + t_{pe}/2)(\Delta F_x - \nu_p \Delta F_y) + 24z_m(z_m + t_{pe}/2)(\bar{F}_x - \nu_p \bar{F}_y)}{E_p t_p^3} \tag{8c}$$

$$(\varepsilon_y)_p^t = \frac{12t_{pe}(z_m + t_{pe}/2)(\Delta F_x - \nu_p \Delta F_y) + 24z_m(z_m + t_{pe}/2)(\bar{F}_x - \nu_p \bar{F}_y)}{E_p t_p^3} \tag{8d}$$

Substituting Eq. (7) into strain-stress relationship, leads to the strains in the piezoelectric actuator at the bottom and top interfaces as follows.

$$(\varepsilon_x)_{pe}^b = \frac{1}{E_{pe}}[(\sigma_x)_{pe}^b - \nu_{pe}(\sigma_y)_{pe}^b] + \varepsilon_{pe}$$
$$= \frac{(\bar{F}_x - \nu_{pe}\bar{F}_y) - 3(\Delta F_x - \nu_{pe}\Delta F_y)}{E_{pe}t_{pe}} + \varepsilon_{pe} \tag{9a}$$

$$(\varepsilon_y)_{pe}^b = \frac{1}{E_{pe}}[(\sigma_y)_{pe}^b - \nu_{pe}(\sigma_x)_{pe}^b] + \varepsilon_{pe}$$
$$= \frac{(\bar{F}_y - \nu_{pe}\bar{F}_x) - 3(\Delta F_y - \nu_{pe}\Delta F_x)}{E_{pe}t_{pe}} + \varepsilon_{pe} \tag{9b}$$

$$(\varepsilon_x)^t_{pe} = \frac{1}{E_{pe}}[(\sigma_x)^t_{pe} - \nu_{pe}(\sigma_y)^t_{pe}] + \varepsilon_{pe} = \frac{(\bar{F}_x - \nu_{pe}\bar{F}_y) + 3(\Delta F_x - \nu_{pe}\Delta F_y)}{E_{pe}t_{pe}} + \varepsilon_{pe} \quad (9c)$$

$$(\varepsilon_y)^t_{pe} = \frac{1}{E_{pe}}[(\sigma_y)^t_{pe} - \nu_{pe}(\sigma_x)^t_{pe}] + \varepsilon_{pe} = \frac{(\bar{F}_y - \nu_{pe}\bar{F}_x) + 3(\Delta F_y - \nu_{pe}\Delta F_x)}{E_{pe}t_{pe}} + \varepsilon_{pe} \quad (9d)$$

The strain continuity conditions at the bottom and top interfaces between the embedded actuator and plate are

$$(\varepsilon_x)^b_p = (\varepsilon_x)^b_{pe} \;;\; (\varepsilon_y)^b_p = (\varepsilon_y)^b_{pe} \;;\; (\varepsilon_x)^t_p = (\varepsilon_x)^t_{pe} \;;\; (\varepsilon_y)^t_p = (\varepsilon_y)^t_{pe} \quad (10)$$

Substituting Eqs. (8) and (9) into Eq.(10), solve for ΔF_x, ΔF_y, $\bar{F}_x$ and $\bar{F}_y$ as follow:

$$\Delta F_x = \Delta F_y = \frac{4E_{pe}^2 t_{pe}^2 z_m(-1+\nu_p)}{[2E_{pe}t_{pe}(t_{pe}^2 + 12z_m^2)(-1+\nu_p) - E_p t_p^3(-1+\nu_{pe})](-1+\nu_{pe})}\varepsilon_{pe} \quad (11a)$$

$$\bar{F}_x = \bar{F}_y = \frac{E_{pe}t_{pe}[2E_{pe}t_{pe}^3(-1+\nu_p) - E_p t_p^3(-1+\nu_{pe})]}{[2E_{pe}t_{pe}(t_{pe}^2 + 12z_m^2)(-1+\nu_p) - E_p t_p^3(-1+\nu_{pe})](-1+\nu_{pe})}\varepsilon_{pe} \quad (11b)$$

Substituting Eq. (11) into Eq. (4), yields to the bending moment

$$m_x = m_y = C_o\varepsilon_{pe} \quad (12a)$$

$$C_o = \frac{2E_p E_{pe} t_p^3 t_{pe} z_m}{-2E_{pe}t_{pe}(t_{pe}^2 + 12z_m)(-1+\nu_p) + E_p t_p^3(-1+\nu_{pe})} \quad (12b)$$

3. Deflection of a simply supported plate

The host plate is a rectangular plate with simply supported boundary conditions. The location of the embedded actuators viewed from the top is shown in Fig. 3. The activated piezoelectric actuators will induce bending moments as derived in Eq. (12) to the plate and can be expressed in terms of unit step functions as follow.

$$m_x = m_y = C_o\varepsilon_{pe}\left[h(x-x_1) - h(x-x_2)\right]\left[h(y-y_1) - h(y-y_2)\right] \quad (13)$$

Using the classical plate theory, the equilibrium equation for the plate can be written in terms of the plate internal moments M_x, M_y, M_{xy} and the actuators induced moments m_x, m_y as

$$\frac{\partial^2(M_x - m_x)}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x\partial y} + \frac{\partial^2(M_y - m_y)}{\partial y^2} = 0 \quad (14)$$

The internal moments M_x, M_y, M_{xy} can be expressed in terms of the flexural displacement w. Moving the moments m_x, m_y to the right hand side of Eq. (14), yields to

$$D\nabla^4 w = \frac{\partial^2 m_x}{\partial x^2} + \frac{\partial^2 m_y}{\partial y^2} \quad D \text{ plate flexural rigidity} \quad (15)$$

Substituting Eq. (13) into Eq. (15), leads to the governing differential equation

$$D\nabla^4 w = C_o\varepsilon_{pe}[\delta'(x-x_1) - \delta'(x-x_2)][h(y-y_1) - h(y-y_2)] + C_o\varepsilon_{pe}[h(x-x_1) - h(x-x_2)][\delta'(y-y_1) - \delta'(y-y_2)] \quad (16)$$

For a simply supported rectangular plate, the flexural displacement w can be expressed as following Fourier series

$$w(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b} \quad (17)$$

Substituting Eq. (17) into Eq. (16), solve for the constant W_{mn} as follow [5].

$$W_{mn} = \frac{P_{mn}}{D[(\frac{m\pi}{a})^2 + (\frac{n\pi}{b})^2]^2} \quad (18a)$$

$$P_{mn} = \frac{4C_o\varepsilon_{pe}}{ab}[-\frac{\gamma_m^2 + \gamma_n^2}{\gamma_m\gamma_n}(\cos\gamma_m x_1 - \cos\gamma_m x_2)(\cos\gamma_n y_1 - \cos\gamma_n y_2)] \quad (18b)$$

$$\gamma_m = \frac{m\pi}{a} \;;\; \gamma_n = \frac{n\pi}{b} \quad (18c)$$

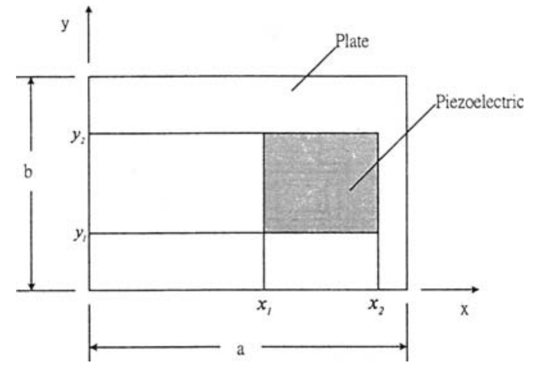

Fig. 3. Geometry of the plate with actuator.

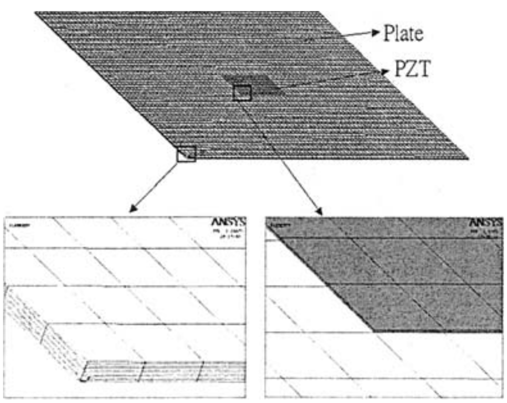

Fig. 4. 3-D finite element mesh.

4. Einite element analysis

The finite element method is a widely used and powerful tool for analyzing complex structures. It is capable of dealing with the piezoelectrical materials. Many researchers have modelled the piezoelectric actuation using the finite element method. A commercially available finite element software ANSYS has the ability to analyze the piezoelectrical materials. In this study, the finite element software ANSYS is adopted to investigate the deflection of a simply supported plate induced by the embedded piezoelectric actuators. To perform the ANSYS finite element analysis for the piezoelectric actuator embedded structure, SOLID 45 elements are used in the host plate and SOLID 5 elements are used for the piezoelectric actuators. The constitutive equations employed for the piezoelectric materials are as follow.

A typical three dimensional finite element mesh is shown in Fig. 4. A voltage between the upper and lower surfaces of the SOLID 5 elements is applied, which results in an electric field along the poling direction of the actuator. The deflections obtained from the finite element method are compared with the analytical solutions of Eq. (17) to validate the present approach.

5. Numerical validation and examples

In the following numerical examples, the host plate is taken as steel with the material properties of Young's modulus $E_p = 207GPa$, Poisson's ratio $\nu_p = 0.292$, density $\rho_p = 7870\ kg/m^3$. The dimensions of the plate are length $a = 0.38\ m$, width $b = 0.3\ m$, thickness $t_p = 1.5876\ mm$. The piezoelectric actuator is assumed to be PZT G-1195 with the material properties of Young's modulus $E_{pe} = 63GPa$, Poisson's ratio $\nu_{pe} = 0.3$, density $\rho_{pe} = 7600\ kg/m^3$, piezoelectric constant $d_{31} = 1.9 \times 10^{-10}\ V/m$ and thickness $t_{pe} = 0.15876\ mm$. The effects of the size and location of the actuators are presented through a parametric study to investigate the deflection and deformed shape of the plate activated by the embedded piezoelectric actuators.

Example 1: Three different sizes of actuators

Two piezoelectric actuators are embedded symmetrically with the midplane of the host plate. The embedded depth is $z_m = \pm 0.55566mm$ measured from the midplane. Three different size of piezoelectric actuators with the dimensions of 0.06 m×0.04 m, 0.08 m×0.06 m and 0.1 m×0.08 m, respectively, embedded in the central area of the host plate as shown in Fig. 5 are considered in this example. The voltages of +1V and −1 V are applied to the upper and lower actuators, respectively, resulting in a bending moment acting on the host plate. The deflections of the host plate induced by the actuators are calculated using both the analytical solution of Eq. (17) and finite element method. Fig. 6 shows the flexural displacements along the horizontal line $y = b/2$ and the vertical line $x = a/2$ of the host plate. The deflection is increasing as the size of actuators increases. The analytical solutions of Eq. (17) of the plate deflection agree well with the finite element method. The maximum deflections in the plate induced by the three different sizes of actuators are listed in Table 1. It shows that the difference between the present approach and finite element method is within 5 %.

Example 2: Three different locations of actuators

In this example, the piezoelectric actuators are embedded in three different locations as shown in Fig. 7, to study the capability of control the deflection shape of the plate by placing the actuators at various locations. The actuators with the dimension of 0.06 m×0.04 m are embedded at the depth of $z_m = \pm 0.55566$ mm measured from the mid-plane of the plate. The flexural displacements of the plate along the horizontal line $y = b/2$ and vertical line $x = a/2$ are shown in Fig. 8. The deflections of the plate obtained by the present approach of Eq. (17) and finite element method are in close agreement. Table 2 lists the maximum deflections of the plate induced by the piezoelectric actuators. The difference of the deflected curve shown in Fig. 8 demonstrates that the shape of the plate can be controlled by placing the actuators at various locations.

6. Conclusions

Piezoelectric materials are often used as strain actuators and active control of structural vibration, as they are compact and have a wide frequency range. In this investigation, two piezoelectric actuators are

Table 1. Maximum deflection (mm) of the plate obtained by ANSYS and Eq. (17) for three different sizes of piezoelectric actuator.

Size \ Method	ANSYS	Eq.(17)	error (%)
PZT 0.06 m×0.04 m	-0.000246	-0.000239	2.84
PZT 0.08 m×0.06 m	-0.000422	-0.000407	3.55
PZT 0.1 m×0.08 m	-0.000616	-0.000585	5.03

Table 2. Maximum deflection (mm) of the plate obtained by ANSYS and Eq. (17) for three different locations of piezoelectric actuator.

Location \ Method	ANSYS	Eq.(17)	error (%)
PZT in central	-0.000246	-0.000239	2.84
PZT at right	-0.000161	-0.000156	3.1
PZT on top	-0.000179	-0.000176	2.08

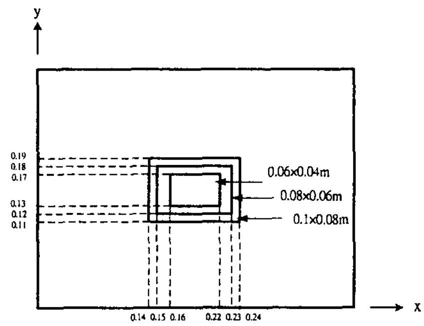

Fig. 5. Three different sizes of actuators.

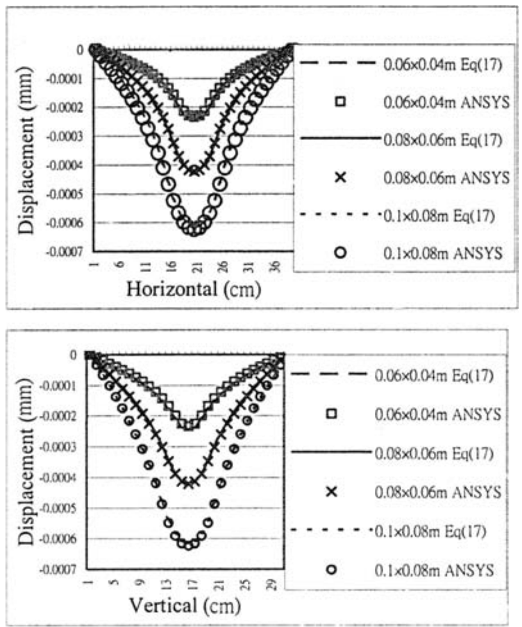

Fig. 6. Flexural displacement of the plate obtained by ANSYS and Eq. (17) along the horizontal line (y=b/2) and vertical line (x=a/2) for three different sizes of actuator.

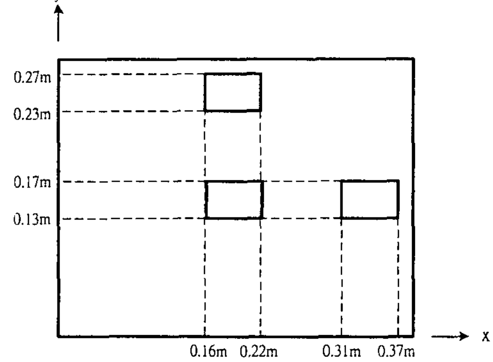

Fig. 7. Three different locations of the actuator.

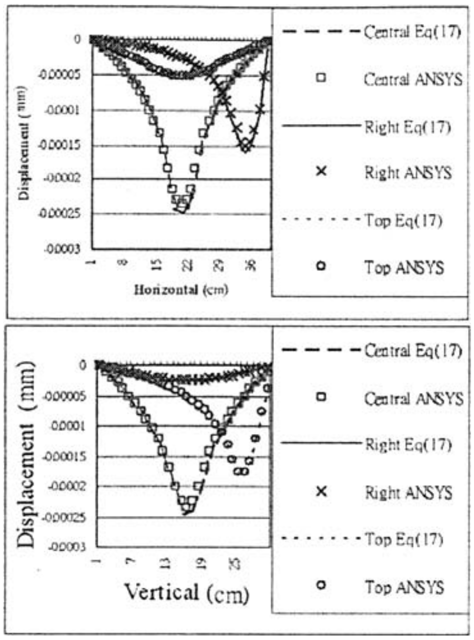

Fig. 8. Flexural displacement of the plate obtained by ANSYS and Eq. (17) along the horizontal line (y=b/2) and vertical line (x=a/2) for three different locations of actuator.

embedded in the plate and are symmetric with the mid-plane. Electric voltages with the same amplitude and opposite sign are applied to the two symmetric piezoelectric actuators, resulting in the bending effect on the plate. Theoretical model of the bending moment is derived by using the theory of elasticity to represent the interaction of the actuator and the host plate. Following the classical plate theory, the deflection of a simply supported plate subjected to the bending moment can be obtained. The analytical solutions are validated with the finite element results. The effects of size and location of actuators on the responses of the plate are presented through parametric study.

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